Walk-through for Math 241 Midterm II.

Work on this preferably together with a partner or when you are in the tutoring room (MTWRF 4-8 p.m, 345 AH) so that the tutors can get you unstuck. Definitely with a pencil in hand and scratch paper so you can scribble.

1 Gradient, Second Derivative Test, Directional Derivatives

1. What is the gradient? Does it make sense to say that the gradient of a two variable function $f(x, y)$ at the point $(2, 3)$ is 4?
2. How does the gradient of a function relate to the partial derivatives of the function?
3. If $f(t)$ is a one variable function, how would you find it's maxima and minima?
4. Now if $f(x, y)$ is a two variable function, how do we find critical points? More specifically, what do we set equal to zero?
5. What determinant do we use to classify critical points? Write down the criteria which indicate maxima, minima, saddles and cases where we don't have enough information. (Hint: the sign of the determinant)
6. What is a closed set? What symbols usually show up with closed sets? Bounded set? Give examples and non-examples?
7. Why do we care about the kinds of sets in the previous question? (Hint: a theorem)
8. Given a non-zero vector, can you construct a vector that is in the same direction as the given one but has length one? What is the name of this object you just constructed?
9. What is a directional derivative? How does the gradient get involved in computing them? Give a geometric way of thinking about it. Why did I place this question right after the previous one?
10. Take your favourite function $f(x, y, z)$ and favorite unit vector $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and compute the directional derivative in this direction at the point $(2, 3, 1)$.
11. (Determine the sign of) / (Estimate the value of) a directional derivative in a given direction from a contour map of f.
12. The partial derivatives of a function $f(x, y)$ are the directional derivatives for which choice of unit vectors?
13. Suppose I am standing on a surface $z = f(x, y)$ at a point $(x, y) = (2, 2)$. Which direction should I move so that f increases as fast as possible? So that f decreases as fast as possible? To initially leave f essentially unchanged?
14. What is the interpretation of the magnitude of the gradient?
15. If I give you a contour map of the function $f(x, y)$ and pick a random point on a contour, can you draw a plausible gradient vector at that point? (Think about what angle it should make with the contour line and whether it should point toward higher or lower contours)
16. Given a surface defined by $f(x, y, z) = k$ and a point P on this, how does the gradient of f at P help in finding the tangent plane at P? In particular which important ingredient of the plane is given by the gradient vector?
17. Same setting as previous problem. Instead of a tangent plane, can you give me the equation of a line NORMAL to the surface and passing through P? (Hint: Gradient helps again.)

2 Lagrange Multipliers

1. What is the method of Lagrange multipliers? Write it out with the relevant assumptions.
2. Suppose I want to find the possible max and min of some function $f(x, y)$ on $\mathbb{R}^2$, can I use Lagrange multipliers?
3. What is the condition $\nabla f = \lambda \nabla g$ geometrically in terms of the intersection of the contours of f and g?
4. Draw several contours of the function $f(x, y) = xy$ (positive, negative, and zero) and the constraint $g(x, y) = x^2 + y^2 = 8$ and find the max and min of f subject to $g = 8$ JUST FROM YOUR PICTURE! No cheating!
5. Draw the contours of $f(x, y) = x^2 + 4y^2$.

3 Parametrization, $r(t)$, dr, ds, $\int_C f ds$, $\int_C F \bullet dr$

1. Given a curve C with a start and an end, if I ask you for a parametrization, what are the two parts of this that you need to write down? In particular, what should you NEVER forget to write?
2. When you are asked to draw a parametrization $r(t)$ on some interval $a \leq t \leq b$, what all detail should your picture involve?
3. Given an equation of the form $f(x, y) = 0$, be able to parametrize it. Give yourself examples of circles and ellipses (of various radii), (clockwise and counterclockwise), (centered around some point not the origin),(Starting and ending at different points). PARAMETRIZE THE HECK OUT OF THESE!
4. Parametrize the part of a hyperbola (Eg: $x^2 - 3y^2 = 1$) where x is negative. How does this change if I ask for the part where x is positive?
5. Given a parametrization $r(t)$ of a curve (NO FUNCTIONS OR FIELDS YET), what is dr? Is it a scalar or a vector? If I say $r(t) = (x(t), y(t))$, now how would you write down dr?
6. What is the difference between a function and a vector field? Give me examples of both. What is the relation between dr and ds?
7. Given a parametrization $r(t)$ (NO FUNCTIONS OR FIELDS YET), what is ds? Is it a scalar or a vector? If I say $r(t) = (x(t), y(t))$, now how would you write down ds?
8. Given a curve C by a parametrization $r(t)$, from $a \leq t \leq b$, what is the formula for it's length? Which of $d\mathbf{r}$ and ds is relevant here?
9. If I give you a FUNCTION $f(x, y)$ and a curve C parametrized by $r(t)$ and asked you to integrate it along a curve, which would you use in the integration, dr or ds? What if I gave you a VECTOR FIELD instead of a function?
10. In the previous problem, if f represents density, what physical meaning does the integral have?
11. Write down your favorite function (Don't make it super simple) and your favorite curve and now integrate this function along that curve. Try not to make a mess.
12. Write down your favorite vector field and your second favorite curve (significantly different from the previous problem please), integrate this field along that curve.
13. Now if $r(t)$ represents the position of a particle, how do I get it's velocity, speed and acceleration? Does this have any connection to question 28?
14. Now go back and think about whether you can interpret all these questions for three dimensions instead of two.

4 Path independance, Simply Connected, Potential Function etc.

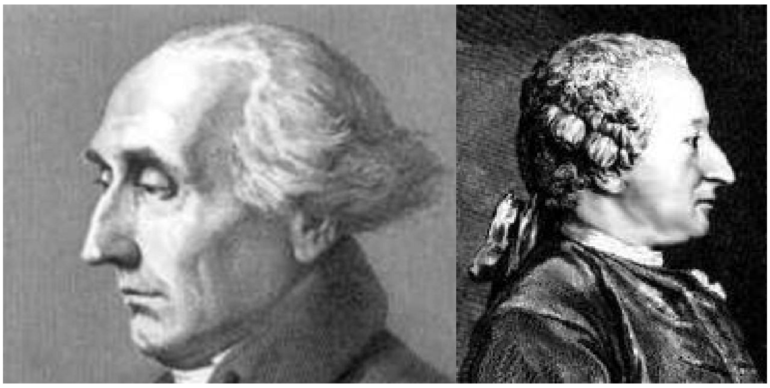
1. Draw the vector fields given by $\langle x, y\rangle$, $\langle y, 0\rangle$ and $\langle y, -x\rangle$. Draw big and don't be stingy with arrows.
2. Look at $\int_C Pdx + Qdy$ where P and Q are functions of two variables. Which kind of integral is this, $\int_C F \bullet dr$ or $\int_C f ds$? Explain how you would get f/F(whichever is correct) from P, Q and vice-versa.
3. What does it mean for a line integral $\int_C F \bullet dr$ to be path independant. Say this aloud till you are no longer confused.
4. Why do we like path independant line integrals?
5. What is a conservative field? (Hint somehow involves the previous two questions)
6. What is the line integral of a conservative field around a loop (i.e. curve beginning and ending at same point)?
7. What is the fundamental theorem of line integrals?
8. What does it mean when we say a vector field F is a gradient field?
9. What is the gradient test to check if a field $F = \langle P, Q\rangle$ is a gradient field? (Hint: Partial derivatives)
10. What is an open set? What symbols usually signify open sets? What is a connected set? A simply connected set? (Hint for last: Holes)

11. If a continuous field F passes the gradient test, what conditions do we need on the domain D it is defined on to guarantee that it is conservative?

12. Find a friend in Math 241 and explain to them the relations between the notions of conservative field, gradient test, path independance.

13. Suppose problem on the exam has a ONLY a picture of some crazy curve C and you are only able to read off the starting and ending points, and they are asking you to compute $\int_C Pdx + Qdy$, for explicitly given functions P, Q. What conditions (On P, Q and the domain) are required so that you would be able to solve this problem?

Here are pictures of Lagrange and Clairaut.



(a) Lagrange (b) Clairaut

Figure 1: The sources of your suffering